COMPUTER MODELING COULOMB'S LAW AND KIRCHHOFF'S LOOP THEOREM THROUGH CONCEPT OF ENERGY

Jozef Hanc, Robert Andrassy

ABSTRACT
Electrostatic fields in space around charged conductors or steady currents in circuits distribute in such way that energy stored in the field or the rate at which energy is dissipated in the circuit is as least as possible. Using computer modeling in the well-known Easy Java Simulations environment we present this alternative, so-called variational, formulation of Coulomb's law and Kirchhoff's loop theorem which demonstrates universality of energy concept and allows students to derive effectively and quickly fields and currents in many common or uncommon physical situations like fields around a sphere, wires and capacitors of different shapes or currents in various types of dc circuits. Advantage of the approach is a development of conceptual understanding and no use of the vector calculus mathematics or additional sign conventions for potential difference typically used in given situations.

KEYWORDS
Coulomb's Law, Kirchhoff's loop theorem, energy, variational principles, computer modeling, Easy Java Simulations

INTRODUCTION

Theories of different physics branches have been stated and reformulated in several, frequently very distinct ways. Probably the most apparent example is a theory of quantum mechanics, which was presented during 20th century at least in nine different pictures (Styer, 2002) – formulations, among them the most well-known are Schrödinger's wave, Heisenberg's matrix and Feynman's amplitude versions of quantum mechanics.

But the same could be stated about classical physics, where classical mechanics in terms of force and momentum concepts were formulated as Newtonian dynamics, or on basis of energy concept or principle of least action as Lagrangian or Hamiltonian dynamics. Similarly geometrical optics can be treated through laws of reflection and refraction, but there also exists a unified approach via Fermat's principle of least time.

In all these cases there is no experimental way to distinguish between formulations of any branch of physics. So the natural question arises: should we care about different formulations of given subject when each provides identical experimental results? There are several reasons why the answer is yes. Here are most important ones (Styer, 2002):

- some problems are difficult to solve in one formulation, but easy in another;
- different formulations have different success, when we attack new phenomena or situations;
- different formulations provide different insights on the same problems, which allows us better understanding and prediction of natural phenomena;

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In 2004 at the Gordon Research Conference[1] dealing with physics research and education in mechanics and at Girep[1] 2006 on modelling in physics and physics education there were a plenary session *Rethinking the mechanics curriculum: action as a unifying theme* (plenary lectures: Taylor, Hanc, Moore) and symposium *Action on stage: ways to unify classical and quantum physics using the action model* (Ogborn et al. 2008, Hanc 2008, Taylor 2008) connected with teaching alternative formulations and models for optics, classical and quantum mechanics using concept of energy and least action principles (Hanc et al. 2003, Hanc & Taylor 2004, Ogborn & Taylor 2005).

This paper is a continuation of that theme and it describes briefly using alternative variational[2] (action) models with computer modeling in electrostatics governing by Coulomb's law and in the field of direct (steady) current circuits analyzed by Kirchhoff's rules: loop rule and junction rule. Concerning computer modeling we decided to use Easy Java Simulations environment whose author is F. Esquembre and it represents a modern, successful and opens source environment for a comfortable and very rapid development of JAVA applets describing explored phenomena. The detailed description of this environment can be found in Christian & Esquembre (2007).

COULOMB'S LAW IN TERMS OF LEAST POTENTIAL ENERGY PRINCIPLE

Traditional approaches. In traditional approach Coulomb's law is postulated as a basic force law of electrostatics that has survived every experimental test. Textbooks (e.g. Halliday et al. 2005; Tipler & Mosca 2003) remind a description of Charles Coulomb's experiment based on the torsion balance and states the inverse-square law for the electric force between two stationary charged particles as a result of this experiment or some textbooks (e.g. Moore 2003, Chabay & Sherwood 2002) only simply postulate this law without any reference to history.

Cummings et al. (2004) besides mentioning Coulomb's experiment pay attention to a real experiment verifying the inverse square relationship with modern tools (digital video camera, video-analysis and data-analysis software) available in many introductory laboratories. Mutual positions of two negatively charged ping-pong balls covered by conducting paint are observed and recorded with the video camera, so the further video-analysis provides data whose fitting leads to a conclusion that force between electrical charges falls off distance proportionally as ($1/r^2$), where a constant of proportionality $k = 8{,}99\times10^9$ N.m^2/C^2 or $k=1/(4\pi\varepsilon_0)$ with the so-called electric constant $\varepsilon_0 = 8{,}85\times10^{-12}$ C^2/(N.m^2).

Approach „between" a simple text stating Coulomb's law and using a real experimental setup for its verifying is presented in Physlets Physics of Christian & Belloni (2004), where in illustration 22.3 students by means of an interactive simulation can explore a force between two point charges. In the simulation a student is able to measure values of force v. distance and to get own plot of data and e.g. in Excel he can try different fits to obtained data points for finding or verifying $1/r^2$.

All these approaches are based on force concept. Our alternative approach based on energy concept and considered as a supplementary to previous ones in a general sense of the introduction of this paper, is motivated by this simple analogy.

Simple mechanical analogy. It is common experience that motion usually slows down and stops and a given mechanical system, e.g. marble falling on the ground, ends in equilibrium with zero kinetic energy. From our experience it is also intuitively clear that a system reaches equilibrium at a local minimum[3] of the potential energy. This result known as *the principle of least potential energy* for systems in equilibrium is a manifestation of conservation of energy. Indeed, if a marble is at rest and has minimal potential energy than any further displacement of it would lead to a positive change in marble's potential energy. According to conservation of energy this positive change would be

[1] <www.grc.org/programs/2004/physres.htm>, <www.girep2006.nl>

[2] Variational principles are formulations of physical laws in the form of minimizing a certain quantity, which is generally called action and which led to required description of phenomena.

[3] Generally it can be a stationary point of potential energy, but only in local minimum static equilibrium is stable.

compensated by a corresponding negative change in kinetic energy. Because the marble is at rest, the negative change in kinetic energy is impossible, so conservation of energy forbids any displacement. Stated another way: equilibrium is a result of conservation of energy.

Variational principle for electrostatic fields. In analogy with mechanics we extend validity of least potential energy principle to electrostatics, particularly we will assume that the real electrostatic field around charged objects keeps such configuration in given volume to minimize its electrostatic energy or in Feynman's words (Feynman et al, 1964) the true electric field is the one with the minimum total energy. Now to illustrate quickly this alternative formulation of electrostatics, we concentrate on a simple example: a field around a sphere (or point charge)[4] in given volume (a larger sphere), which gives Coulomb's law.

Fundamental concepts of our approach are energy concepts closely connected with electric field $\boldsymbol{E}$: energy density u of any electrostatic field needed to calculate the total energy of the field and electric potential φ for calculating electric field $\boldsymbol{E}$:

$$u = \frac{\varepsilon_0}{2} E^2 \quad \text{and} \quad \Delta\varphi = -\boldsymbol{E} \cdot \Delta\boldsymbol{r} \qquad (1)$$

These expressions (or definitions) can be taken from standard instruction from textbooks mentioned above or via dimensional analysis or simple reasoning based on conservation of energy.

Computer modeling. Now how to do required minimizing of energy to find the true field around the sphere (Figure 1, left)? One of mathematically easiest ways with no differential, integral or variational calculus, with a support of conceptual understanding, is interactive computer modeling. Figure 2 shows two screens of an interactive program created in Easy Java Simulations environment by authors[5]. The goal of hands-on computer activity using the software is to find such potential function $\varphi = \varphi(r)$ in given volume (larger sphere around smaller charged sphere) for which energy is as least as possible.

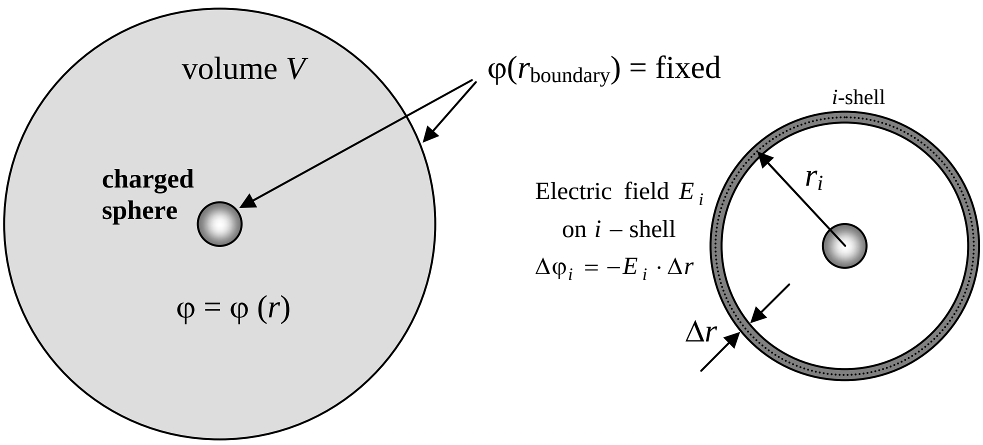


Figure 1. The true electrostatic field in a given spherical volume V subject to boundary conditions, $\varphi(r_{\text{boundary}})$ = fixed, is described by such potential $\varphi = \varphi(r)$ which minimizes energy of electric field.

To see how computer calculates the total energy of a field generated by a trial potential, we remark that the charge distribution on the sphere possesses spherical symmetry, so electric potential describing electric field around the sphere must have spherical symmetry too. Hence potential must depend only on distance from center of the sphere, not on direction or in other words φ is only function of distance r.

[4] This analysis can be also used in other examples like wires and capacitors of different shapes.
[5] Idea of modelling is so simple that it can be programmed in EJS or Excel very quickly by students.

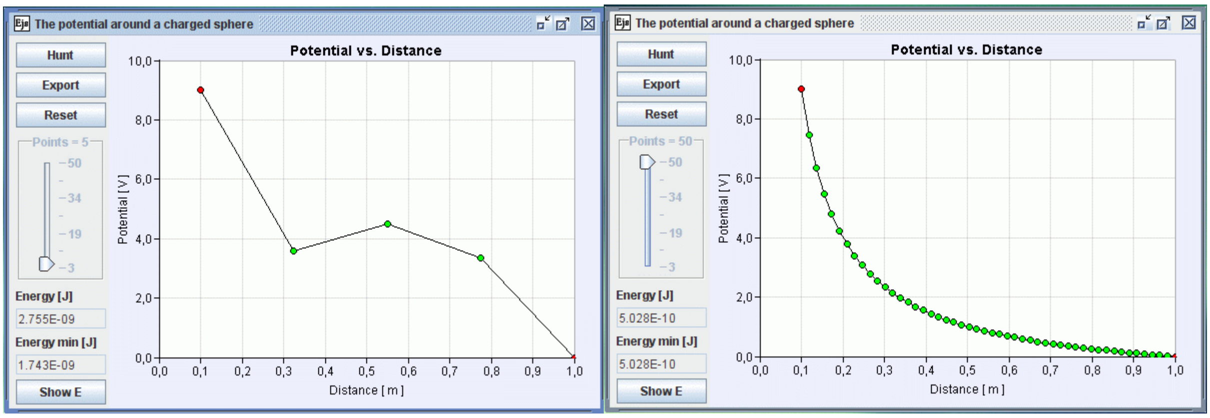


Figure 2. Trial potential function φ = φ (*r*) is represented by connected straight-line segments whose endpoints have first coordinates in equal distance Δ*r* from each other. The user can drag intermediate points (green points) up and down, one at time, to minimize energy of a field generated by given trial potential function. The user can also add many intermediate points to increase the accuracy, and ask the computer (*Hunt button*) to minimize energy by same successive individual dragging of points, which is done in a split second.

Due to spherical symmetry and straight-line-segments representation of potential the volume in which we find the true field is divided into concentric spherical shells with thickness Δ*r*. In each spherical shell electric field can be easily calculated as $E = -\Delta\varphi/\Delta r$, so it has a constant magnitude in each shell inside spherical volume *V* (Figure 1, right). Since according to (1) energy density *u* also appears constant in each shell, the total field energy in volume *V* for a given trial potential is simply sum of energies for all spherical shells

$$\begin{pmatrix}\text{total electrostatic}\\ \text{energy}\end{pmatrix} = \sum_{\substack{\text{all shells}\\ \text{in volume } V}} u_i \Delta V_i \quad \text{, where } u_i = \frac{\varepsilon_0}{2}\left(\frac{\Delta\varphi}{\Delta r}\right)^2_{i\text{-shell}} \text{ and } \Delta V_i = 4\pi r_i^2 \Delta r\,. \qquad (2)$$

Software also provides an export of actively obtained data for spread sheets (Excel, Calc), which can be analyzed to any desired level of detail. In our case through different type of fitting students can find fits of data with increasing number of segments and make final conclusions about true relation between potential and *r* and generated electric field and *r* (Figure 3). Students also verify that relations ($1/r$) and ($1/r^2$) give excellent fits to the obtained potential and field data. It is worth to know that from the values of minimal energy student can also calculate value of capacity of a given system.

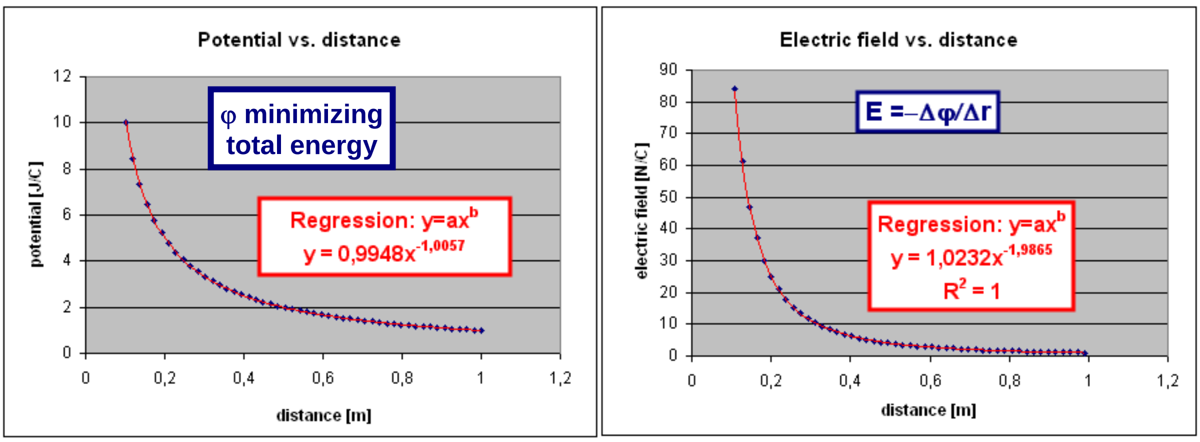


Figure 3. Actively obtained data for true potential and electric field can be copied into spread sheet programs for finding relations: potential v. distance, field v. distance.

KIRCHHOFF'S LOOP THEOREM IN TERMS OF GENERALIZED LEAST ENERGY DISSIPATION PRINCIPLE

Traditional approach. Solving dc circuits, i.e. finding steady currents flowing in arbitrary dc circuits composed of electromotive forces (emfs) and resistors, is traditionally taught by using Kirchhoff's rules. Kirchhoff's first rule, stating that the currents flowing into and out of any junction of a circuit sum to zero, is nothing else as a manifestation of conservation of charge; Kirchhoff's second rule, stating that the potential differences encountered in traversing any closed loop of a circuit sum to zero, is essentially principle of conservation of energy on per unit charge basis. Application of the rules to a typical circuit gives a system of simultaneous linear equations to be solved by students for finding the unknown steady currents in the circuit.

Simple mechanical analogy. In introducing our alternative energy approach, using ideas of Van Baak (1999), we will again consider a simple mechanical analogy for motivating and postulating variational principle describing dc circuits with resistors and batteries[6]. Imagine again a marble, but now falling in water and assume that the marble is in „steady state", i.e. the marble reached its terminal velocity v.

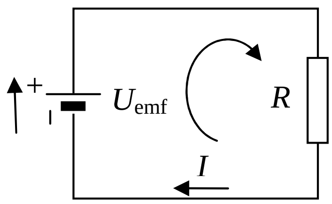
When we analyze uniform motion of the marble during very small time increment Δt, then gravity mg acting downward on the marble, in direction of motion, increase kinetic energy by amount equal to work mg $(v\Delta t)$, where $(v\Delta t)$ is an increment displacement of the marble during Δt. The opposite drag (viscous) force bv has a tendency to decrease marble's kinetic energy by bv $(v\Delta t)$. Since marble's kinetic energy is constant during motion, the principle of conservation of energy tells us that both changes in kinetic energy are equal $bv(v\Delta t) = mg(v\Delta t)$, which dividing by displacement $(v\Delta t)$ gives correct Newton's second law with zero acceleration. Notice that dividing by Δt (in other words considering works per second or powers) we have another form of energy conservation on per unit time basis

$$bv.v - mg\,.v = 0\,. \tag{3}$$

Now introduce formally the following special combination of the powers of acting forces called power function

$$\begin{pmatrix}\text{power}\\ \text{function}\end{pmatrix} \equiv \frac{1}{2}P_{\text{drag}} - P_{\text{gravity}} = \frac{1}{2}bv^2 - mg\,v\,. \tag{4}$$

Then by minimizing[7] the power function with respect to terminal velocity v we get energy conservation equation (3). Stated another way, as a variational principle: *in steady state a marble reaches such terminal velocity which leads to the minimal power function.*



Variational principle for dc circuits. Now try to find analogy between a marble falling in water and a simple circuit consisting of battery, a source of constant terminal voltage U_{emf}, and a resistor with resistance R. Assuming steady current I flowing in the circuit, we analyze it from energy viewpoint, but for comparison of results we also introduce a loop.

During a very small time increment Δt, charge $\Delta Q = I\Delta t$ moves between terminals of the battery and therefore it obtains potential energy U_{emf} $(I\Delta t)$. It means that battery supplies energy to the system in similar way as gravity force in case of the falling marble. At the same time resistor R dissipates energy

[6] We will consider ideal dc circuits whose detailed description is given in Cummings et al (2004), part 3, p. 773 or Van Baak (1999).
[7] It can be done using elementary differential calculus setting derivative of the power function to zero or by simple high school geometrical considerations about parabola using its geometrical symmetry, from which results that vertex of parabola lies exactly in the middle between intersection points of parabola with x-axis (roots of quadratic equation).

$RI^2\Delta t$, well-known Joule heating. (It is similar to dissipation of marble's energy in water.) Hence the principle of conservation of energy tells us that increase in potential energy due to the battery must equal heat generated in the resistor: $RI^2\Delta t - U_{\text{emf}} I\Delta t = 0$, which dividing by $I\Delta t$ gives Kirchhoff's second rule for our circuit based on displayed loop. Notice that dividing by Δt we again get energy conservation on per unit time basis analogical to equation (3):

$$RI^2 - U_{\text{emf}} I = 0 \,. \tag{5}$$

Summarizing previous considerations we see that current I corresponds to marble's velocity v, expression RI corresponds exactly with viscous force bv and U_{emf} is the „force" applied by battery analogical to gravity mg. To finish our analogy we constitute power circuit function

$$\begin{pmatrix}\text{power}\\ \text{function}\end{pmatrix} = \frac{1}{2}P_{\text{res}} - P_{\text{emf}} \,, \tag{6}$$

where P_{res} is the dissipation in the resistor, P_{emf} is the rate at which U_{emf} is doing work on the current I that is flowing through it. Then true current I is that one, which minimizes power circuit function – the difference (6).

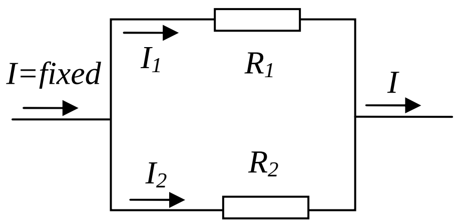
The previous result can be generalized to any network of Ohmic resistors and ideal emfs (Van Baak, 1999), where all the internal steady currents distribute in such way that the difference

$$\begin{pmatrix}\text{power}\\ \text{function}\end{pmatrix} = \frac{1}{2}P_{\text{res}} - P_{\text{emf}} = \frac{1}{2}\sum_{\substack{\text{all}\\ \text{resistors } R_k}} R_k I_k^2 - \sum_{\substack{\text{all}\\ \text{emf } U_j}} U_j I_j \tag{7}$$

is as little as possible, whereby currents in all circuit junctions must satisfied conservation of charge. We remark that if some current I_j flows in opposite direction to the direction of some emf U_j, then energy transfer is from charge carries to the battery and rate at which emf U_j is doing work on the current I_j must be taken positive in (7).

Computer modelling. Minimizing with respect to all currents can be done again by means of computer modelling. The algorithm demonstrated in Figure 4 is principally identical with the algorithm used for finding the true electric field in previous section. Only difference consists in auxiliary condition, conservation of charge, meaning that not all currents are independent and to minimize power function we need to change only independent currents.

Finally we remark that for dc circuits we can consider situations without emfs (see Van Baak, 1999) like two resistors conected in parallel, Wheatstone combination of five resistors or cube combination of twelve resistors. In this special case we have



$$\begin{pmatrix}\text{power}\\ \text{function}\end{pmatrix} = \frac{1}{2}\sum_{\substack{\text{all}\\ \text{resistors } R_k}} R_k I_k^2 \,,$$

whose minimizing is equivalent to minimizing the total dissipation $\sum R_k I_k^2$ in the circuit. Therefore we call the principle (7) generalized least energy dissipation principle. It is worth to know that the single equivalent resistance to given network can be found directly from minimum value of total dissipation.

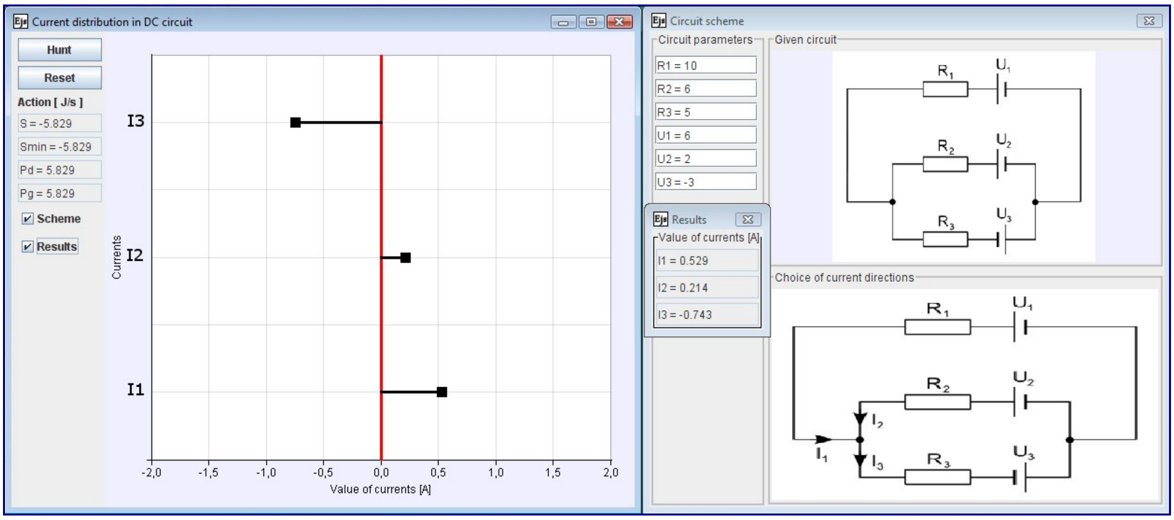


Figure 4. Screen shot from EJS simulation for particular dc circuit. The user can move with currents to the left or to the right to find such values of the currents, which minimize power circuit function (also called "action"); the user can read values of required currents directly from *x*-coordinates or they can be displayed in a window with numerical results. In our example minimizing leads to values I_1= 0,529 A, I_2 = 0,214 A, I_3 = – 0,743 A.

CONCLUSIONS

Presented variational minimization via computer modeling and leading to finding true potential or true distribution of currents are applying the algorithm described principally in Feynman et al[8] (1964) and used in mechanics, optics, relativity by Hanc et al. (2003), Hanc & Taylor (2004), Hanc et al (2005) and Taylor (2008). A critic might object that the minimizing procedures via automatic computer minimizing is too easy, since all the student does is push a few buttons. But with this tool the student can investigate interactively and effectively much more expanded world of possible problems.

From didactical viewpoint supplementing traditional methods of teaching Coulomb's law and Kirchhoff's rules by presented alternative variational approaches provides:

- alternative view of basic laws in electricity, which is important in connection to new situations – e.g. quantum electrodynamics explaining Coulomb's law can start just from least potential energy principle rewritten as the principle of least action (see Feynman & Hibbs 1965), whereas the traditional force concept approach is inconvenient in modern physics
- conceptually simple algorithm of minimization without using of differential, integral or variational calculus
- fact that finding minimized quantities is not abstract theoretical thing, but really quite practical issue because from minimal energy of field we can calculate capacity of given system or from minimal dissipation single equivalent resistance of given network
- considering no loops, no additional separate conventions of signs in circuits leading not to confusing without memorization and decreasing number of student errors in analyzing circuits
- unifying, simpler and more economical approach for finding fields, capacities, currents, resistances without memorizing special rules for calculation of resistance or capacity. For example in case of dc circuits all required quantities and relations between them can be obtain from one principle implicitly including all equations which would result from Kirchhoff's rules.

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[8] Here is the only difference with Feynman's idea for minimizing, all other steps are same. Instead of straight-line segments he uses approximation via polynomial functions.

- modeling which is not as using a black box; students can write actively a computer program where writing formulas is practically identical with standard physical notation
- implicit introduction to computer physics dealing with boundary problems and auxiliary conditions; minimizing approach can be used also for nonsymmetrical situations and today belongs to most powerful numerical tool of engineers (so-called the finite element method)

The main disadvantages of our approach is need:
- for more time if we want to show alternative views together with traditional ones.
- to accept by students validity and more abstract form of power function used in circuits analysis.
- to use computer and to learn EJS environment
- to study the approach not only by students, but also by teachers, since the approaches are not familiar.

To eliminate mentioned disadvantages we recommend strategies summarized in these notes:
- save your time by presenting variational approaches as additional, special self-study projects intended for the best students
- familiarize abstract form of "actions" by analogies mentioned here and by analyzing more examples and comparing with results from traditional approaches
- use free Easy Java simulations also in other branches of physics (also during traditional instruction), since computer modeling is now very important part of teaching; if no, introduced computer modeling can be implemented in Excel
- in calculus based courses computer modeling can be partially replaced by using standard calculus if students have good mathematical background.

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